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ON THE CONVEX FUNCTIONALS ANALOGOUS TO THE GENERALIZED TREFFTZ FUNCTIONALS IN VARIATIONAL PROBLEMS OF THE THEORY OF ELASTICITY *

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A possible general approach for constructing convex functionals in the variational problems of the theory of elasticity, utilizing the Lagrange's multipliers, is indicated. It is shown that such functionals can be constructed in exactly the same manner as the generalized Trefftz functionals which were constructed in /1/ for the basic boundary value problems of the theory of elasticity. The problem of minimizing such functionals using the concept of duality is also discussed.

In /1/ the author constructed, for the basic boundary value problems of the theory of elasticity, the generalized Trefftz functionals. In the case of e.g. the second boundary value problem of the theory of elasticity, these functionals have the form

$$\Phi(\mathbf{u}^*) = 2\int_G W(\mathbf{u}^*)\,dG + \frac{1}{\alpha}[\alpha\|\mathbf{u}^*\|_{1/2,S} + \|\mathbf{t}^{(\nu)}(\mathbf{u}^*)\|_{-1/2,S}]^2 - \alpha\|\mathbf{u}^*\|^2_{1/2,S} \tag{1}$$

$$\|\mathbf{t}^{(\nu)}(\mathbf{u}^*)\|_{-1/2,S} = \sup_{\mathbf{u}^*\in W_2^{1/2}(S)} \frac{|\langle \mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*)\rangle|}{\|\mathbf{u}^*\|_{1/2,S}}$$

Here $\mathbf{u}^*(x)$, $x\in\bar G$ are solutions, in the usual or generalized sense, of the equations $A\mathbf{u}^* = \mathbf{K}(x)$. Every one of these solutions,as was shown in /1/, can be written in the form of a sum $\mathbf{u}^*(x) = \mathbf{u}_0(x) + \varphi_0(x)$, where $\mathbf{u}_0$ is the energy solution of the basic boundary value problem of the theory of elasticity and φ_0 is a solution of the auxilliary problem

$$A\varphi_0 = 0,\quad \mathbf{t}^{(\nu)}(\varphi_0)|_S = \mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S;\quad \bar G = G\cup S \tag{2}$$

$G\subset E_3$ is a bounded region with sufficiently smooth boundary S, occupied by the elastic medium, A is the differential operator of the anisotropic theory of elasticity /2/, $W(\mathbf{u}^*)$ is a positive definite quadratic form relative to the components of the elastic deformation tensor /2/, $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ is the stress vector acting on the surface S, angular brackets denote a bilinear form defined on the dual pair $W_2^{1/2}(S)\times W_2^{-1/2}(S)$ of the spaces of traces of functions belonging to the Sobolev space $W_2^1(G)$, $W_2^{1/2}(S)$ is a Sobolev—Slobodetskii /3,4/ space and $W_2^{-1/2}(S)$ is its conjugate. The norms in the latter spaces are denoted, respectively, by $\|\cdot\|_{1/2,S}$, $\|\cdot\|_{-1/2,S}$.

The following properties of the quadratic functional $\Phi(\mathbf{u}^*)$ are given in /1/:

(a) $$\Phi(\mathbf{u}^*) \geqslant c(\alpha)\|\mathbf{u}^*\|^2_{W_2^1(G)}$$

Here $c(\alpha) > 0$ is a constant depending on a sufficiently small, positive constant α appearing in (1).

(b) When $\mathbf{u}^* = \mathbf{u}_0$ $\Phi(\mathbf{u}^*) = \Phi(\mathbf{u}_0) = |\mathbf{u}_0|^2$, where $|\cdot|$ is the energy norm of the second boundary value problem of the theory of elasticity /2/.

(c) The bilinear functional $\Phi(\mathbf{u}_0, \varphi_0) = 0$.

The above properties imply that the problem of minimizing the functional $\Phi(\mathbf{u}^*)$ has a solution, and we have $\min\Phi(\mathbf{u}^*) = |\mathbf{u}_0|^2$.

Below we show that the construction of the functional $\Phi(\mathbf{u}^*)$ is based on the process of constructing convex functionals in the problems of optimization, with the help of the Lagrange's multipliers. The vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ plays the part of the Lagrange's multiplier in /1/.

Let us find the set U of vectors $\mathbf{u}\in W_2^1(G)$ such that $\mathbf{u}|_S\in W_2^{1/2}(S)$, $\|\mathbf{u}\|_{1/2,S}\leqslant 1$, for which the numerical function

$$f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) = \langle \mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle - \|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2,S} \leqslant 0,\quad \forall \mathbf{t}^{(\nu)}(\mathbf{v})\in W_2^{-1/2}(S)$$

The numerical function $f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v}))$ defined on the dual pair $W_2^{1/2}(S)\times W_2^{-1/2}(S)$ is positively homogeneous in $\mathbf{t}^{(\nu)}(\mathbf{v})$, i.e.

$$f(\mathbf{u}, \lambda\mathbf{t}^{(\nu)}(\mathbf{v})) = \lambda f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})),\quad \forall\lambda\geqslant 0$$

Lemma 1. The set

$$U = \{\mathbf{u}\,|\,\mathbf{u}\in W_2^1(G),\ f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v}))\leqslant 0,\ \forall \mathbf{t}^{(\nu)}(\mathbf{v})\in W_2^{-1/2}(S)\}$$

is a closed convex subspace of the space $W_2^1(G)$. The convexity of the set U is obvious, and its closure follows from the continuity of the mapping $\mathbf{u}\to\mathbf{u}|_S$: $W_2^1(G)\to W_2^{1/2}(S)$.

Lemma 2. The function $f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v}))$ has the property

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{v})\in W_2^{-1/2}(S)} |f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v}))| = \begin{cases} +\infty, & \mathbf{u}\notin U \\ 0, & \mathbf{u}\in U\end{cases}$$

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If $\mathbf{u} \in U$, then $f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) \leq 0$ and it is sufficient to choose $\mathbf{t}^{(\nu)}(\mathbf{v}) = 0$ for the following equation to hold:

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)} f(u, \mathbf{t}^{(\nu)}(\mathbf{v})) = 0$$

If $\mathbf{u} \notin U$, then there exists a vector $\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)$, such that $f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) > 0$, and since $\mathbf{u} \notin U$, $\|\mathbf{u}\|_{1/2, S} > 1$, we have

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)} f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) = \sup_{\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)} [\langle \mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle - \|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2, S}] =$$
$$\|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2, S} \cdot (\|\mathbf{u}\|_{1/2, S} - 1) > 0$$

Then /6/

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)} f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) \geq f(\mathbf{u}, \lambda \mathbf{t}^{(\nu)}(\mathbf{v})), \quad \forall \lambda > 0$$

and hence

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)} f(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})) = +\infty$$

Let us now seek $\inf_{\mathbf{u}^* \in U} I(\mathbf{u}^*)$ where $I(\mathbf{u}^*)$ is a strictly convex functional. Since $U \subset W_2^1(G)$ is a closed convex subset, the problem has a unique solution and can be replaced, by virtue of Lemma 2, by an equivalent problem in which inf is searched for over the whole space $W_2^1(G)$

$$\inf_{\mathbf{u}^* \in W_2^1(G)} \sup_{\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in W_2^{-1/2}(S)} \{I(\mathbf{u}^*) + f(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*))\} = \inf_{\mathbf{u}^* \in W_2^1(G)} \begin{cases} +\infty, & \mathbf{u}^* \notin U \\ I(\mathbf{u}^*) = \inf_{\mathbf{u}^* \in U} I(\mathbf{u}^*) \end{cases} \quad (3)$$

Moreover, if $\mathbf{u}^* \in U$, then $\|\mathbf{u}^*\|_{1/2, S} \leq 1$, and by virtue of the imbedding theorems for the Sobolev–Slobodetskii spaces /4/, we have $\|\mathbf{t}^{(\nu)}(\mathbf{u}^*)\|_{-1/2, S} \leq 1$. Therefore the numerical function

$$\sup_{\|\mathbf{t}^{(\nu)}(\mathbf{u}^*)\|_{-1/2, S} \leq 1} f(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*)) = \|\mathbf{u}^*\|_{1/2, S} - \|\mathbf{t}^{(\nu)}(\mathbf{u}^*)\|_{-1/2, S}$$

has form identical to the term within the square brackets in the expression (1) for the generalized Trefftz functional.

The problem of determining

$$\sup_{\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in W_2^{-1/2}(S)} \inf_{\mathbf{u}^* \in W_2^1(G)} \{I(\mathbf{u}^*) + f(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*))\}$$

is dual to problem (3), and the vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ plays the part of the Lagrange's multiplier /6/.

Thus the construction of the generalized Trefftz functionals obtained in /1/ is identical to the construction of the Lagrangian $L(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*)) = I(\mathbf{u}^*) + f(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*))$ where the form of the function $f(\mathbf{u}^*, \mathbf{t}^{(\nu)}(\mathbf{u}^*))$ is determined, in general, not uniquely.

The generalized Euler-Lagrange equation for the functional $\Phi(\mathbf{u}^*)$ with respect to the vectors $\mathbf{u}_0$ and φ_0, has the form

$$\Phi(\mathbf{u}_0, \varphi_0) = 2\int_G W(\mathbf{u}_0, \varphi_0)\, dG - \int_S \mathbf{u}_0 \cdot \mathbf{t}^{(\nu)}(\varphi_0)\, ds = 0 \quad (\mathbf{t}^{(\nu)}(\mathbf{u}_0)|_S = 0)$$

and coincides, for $\mathbf{u} = \mathbf{u}_0$, with the integral identity determining a generalised solution $\varphi_0 \in W_2^1(G)$ of the auxilliary problem (2)

$$2\int_G W(\varphi_0, \mathbf{u})\, dG - \int_S \mathbf{u} \cdot \mathbf{t}^{(\nu)}(\mathbf{u}^*)\, ds = 0, \quad \forall \mathbf{u} \in W_2^1(G)$$

Thus the vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ playing the part of the Lagrange's multiplier in the expression for the functional $\Phi(\mathbf{u}^*)$, represents a boundary value of the vector $\mathbf{t}^{(\nu)}(\varphi_0)$ in the auxilliary problem (2).

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